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~~Quiver varieties~~ Instantons and double affine Grassmannian

Review of geometric Satake correspondence

G : reductive grp / $\mathbb{C}$

$\mathcal{K} = \mathbb{C}((s)) \supset \mathcal{O} = \mathbb{C}[[s]]$

$Gr_G = G(\mathcal{K})/G(\mathcal{O})$: affine Grassmannian

- ∞-dimensional variety
- $Gr_G \underset{\text{homotopic}}{\simeq} \Omega G_{cpt}$ based loops

$G(\mathcal{O})$-orbits on Gr_G

$\leftrightarrow \lambda \in \Lambda^+$: dominant <u>coweights</u>

$\Lambda^+ \subset \Lambda =$ coweight lattice of $G = \mathrm{Hom}(\mathbb{G}_m, \underset{\text{maximal torus}}{T}) \subset G(\mathcal{K})$

$\cong$ weight lattice of ${}^LG$

$\lambda \in \Lambda^+ \leftrightarrow$ dominant weight of ${}^LG \leftrightarrow$ f.d. irr. rep $V(\lambda)$ of ${}^LG$

$$Gr_G = \bigsqcup_{\lambda \in \Lambda^+} Gr_G^\lambda$$: stratification (analog of Schubert cells)

closure $\overline{Gr_G^\lambda}$: finite dimensional projective variety
usually singular

$IC(\overline{Gr_G^\lambda})$: intersection cohomology complex of $\overline{Gr_G^\lambda}$ (Goresky-MacPherson)
(not sheaf, cpx of constructible sheaves)

(extend $\mathbb{C}_{Gr_G^\lambda}$ rather nontrivial way to $\overline{Gr_G^\lambda}$ so that Poincaré duality holds

$\mathcal{P} = \mathrm{Perv}_{G(\mathcal{O})} Gr_G$: abelian category of $G(\mathcal{O})$-equiv perv. sheaves on Gr

It has a tensor structure via "convolution diagram"

$$G(K) \times_{G(\mathcal{O})} Gr_G = Gr_G \tilde{\times} Gr_G \xrightarrow{\varpi} Gr_G$$

Gr_G-b'dle over Gr_G

$$A * B := \varpi_*(A \tilde{\boxtimes} B)$$

Th. (Lusztig, Ginzburg, Beilinson-Drinfeld, Mirković-Vilonen)

$(\mathcal{P}, *) \cong (\mathrm{Rep}(G^\vee), \otimes)$ as $\otimes$-categories

s.t. $\quad \mathbb{H}^*(IC(\overline{Gr_G^\lambda})) \cong V(\lambda)$

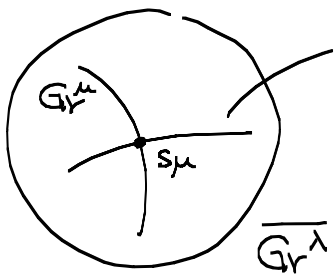
highest weight representation

How about weight space ?

$V(\lambda)_\mu$: weight space $=$ stalk of $IC(\overline{Gr_G^\lambda})$ at $s^\mu \in Gr_G^\mu$

This is the starting point of the geometric Langlands.

- more suitable for double affine generalization



$\overline{M}^\lambda_\mu$: transversal slice to Gr_G^μ in $\overline{Gr_G^\lambda}$

$$\overline{M}^\lambda_\mu \cap Gr_G^\lambda =: M^\lambda_\mu \underset{\text{open}}{\subset} \overline{M}^\lambda_\mu$$

$$V(\lambda)_\mu \cong IC(\overline{Gr^\lambda}) \text{ at } s^\mu \cong IC(\overline{M}^\lambda_\mu) \text{ at } s^\mu$$

<u>Question</u>

What is the affine analog of the affine Grassmann
= double affine Grassmann ?

$V(\lambda)$: ∞-dimensional
$V(\lambda) \otimes V(\mu)$: ∞-direct sum of $V(\nu)$'s

Consider only integrable highest weight rep.
$\Rightarrow$ controllable ∞ sum !

Affine Lie algebra

- $\mathfrak{g}$: simple Lie algebra $/ \mathbb{C}$
- $L\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$: loop algebra
- $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c$: central extension
  c : central
  $[X \otimes t^n, Y \otimes t^m] = [X, Y] \otimes t^{m+n} + m\delta_{m+n,0}(X|Y)c$
- $\mathfrak{g}_{aff} = \hat{\mathfrak{g}} \oplus \mathbb{C}d$ $\quad d$: degree operator
  $d(X \otimes t^n) = n X \otimes t^n$

Kac developed
representation theory of integrable highest weight representations.
related to — modular forms
— CFT

parametrization : Λ^+_{aff} : dominant integral weights
$\lambda \in \Lambda^+_{aff} \rightsquigarrow V(\lambda)$: corresponding representation
$\mu \in \Lambda_{aff}$ $\quad V(\lambda)_\mu$: weight space (finite dim.)

- c acts by the scalar $\langle \lambda, c \rangle$ (positive integer)
  $=$ level
- $\langle \lambda, d \rangle \in \mathbb{Z}$: irrelevant parameter
  $\lambda, \lambda' \in \Lambda^+_{aff}$ s.t. $\lambda \equiv \lambda'$ on $\mathfrak{g}_{aff} / \mathbb{C}d$
  $\begin{pmatrix} \lambda - \lambda' \\ \in \mathbb{C}\delta \end{pmatrix} \Rightarrow V(\lambda) \cong V(\lambda')$ as rep's of $\hat{\mathfrak{g}}$
- $\lambda|_{\mathfrak{g}} \in \Lambda^+$: dominant weight for finite dim'l LA $\mathfrak{g}$.

Conversely if $\lambda_{fin} \in \Lambda^+$, level l, $\langle \lambda, d \rangle$ are given
s.t. $\langle \lambda_{fin}, \text{highest coroot} \rangle \leq l$
$\Rightarrow \exists 1\ \lambda \in \Lambda^+_{aff}$

Check

$$\lambda = \lambda_0 \Lambda_0 + \cdots + \lambda_n \Lambda_n$$

$$l = \langle \lambda_{\text{aff}}, \underset{\underset{a_0\alpha_0 + \cdots + a_n\alpha_n}{\parallel}}{c} \rangle = a_0\lambda_0 + \overbrace{a_1\lambda_1 + \cdots + a_n\lambda_n}^{= \langle \lambda, \text{highest coroot} \rangle}$$

But geometric side: $Gr_{G_{aff}} = G_{aff}(K)/G_{aff}(\mathcal{O})$

and orbits are highly ∞-dimensional!

difficult to define IC sheaves ......

<u>Proposal</u> (Braverman - Finkelberg 0711.2083)

analog of $\overline{M}^\lambda_\mu$ = Uhlenbeck partial cptification of G-instantons on $\mathbb{R}^4/\mathbb{Z}_\ell$

ℓ = level of the rep. of aff. KM group

- $H^*(IC(\text{analog of } \overline{M}^\lambda_\mu)) \cong V(\lambda)_\mu$ rep. of $(G_{aff})^\vee$
- certain diagram $\leftarrow\text{-}\text{-}\text{-}\rightarrow$ $\otimes$
  explained later

G : simple & simply-connected

$\mathrm{Bun}_G^k(\mathbb{C}^2)$ = framed moduli space of G_{cpt}-instantons on S^4 with $c_2 = k$ trivialization at ∞

= framed moduli space of algebraic G-bdles on $\mathbb{CP}^2$ trivialization at $\ell_\infty \subset \mathbb{CP}^2$

smooth of dim $= 2kh^\vee$

$$\mathrm{Bun}_G^k(\mathbb{C}^2) \subset \mathcal{U}_G^k(\mathbb{C}^2) := \bigsqcup_{0 \leq k' \leq k} \mathrm{Bun}_G^{k'}(\mathbb{C}^2) \times S^{k-k'}\mathbb{C}^2$$

Uhlenbeck partial cptification

Fix a hom $\bar{\mu} : \mathbb{Z}_\ell \to G$

$$\mathbb{Z}_\ell \subset SL(2) \subset GL(2)$$

$$\mathbb{Z}_\ell \curvearrowright \mathrm{Bun}_G^k(\mathbb{C}^2) \subset \mathcal{U}_G^k(\mathbb{C}^2)$$

through the action of diagonal emb. to $(\mathrm{incl} \times \bar{\mu}) : \mathbb{Z}_\ell \to GL(2) \times G$

fixed pts $=: \mathrm{Bun}_{G,\bar{\mu}}^k(\mathbb{C}^2/\mathbb{Z}_\ell)$

another inv. $\lambda : \mathbb{Z}_\ell \longrightarrow G$ hom.

$\mathrm{Bun}_{G,\bar\mu}^{\bar\lambda,k}$ = fixed pt set action corr. to the fiber at $0 \in \mathbb{C}^2$

$\mathcal{U}_{G,\bar\mu}^{\bar\lambda,k}$:= ~~fixed pt set~~ closure of $\mathrm{Bun}_{G,\bar\mu}^{\bar\lambda,k}$ in $\mathcal{U}_G^k(\mathbb{C}^2)$

Technical conjecture

$\mathrm{Bun}\ \cancel{\mathcal{U}}_{G,\bar\mu}^{\bar\lambda,k}$: irreducible

Lemma (BF)

$\bar\lambda, \bar\mu \in \mathrm{Hom}(\mathbb{Z}_\ell, G)/\text{conj.} \xleftrightarrow{\text{bijection}}$ level ℓ wts of $\hat{\mathfrak{g}}^\vee$

$\hat{\mathfrak{g}}^\vee$ does not contain the degree operator d

lifts to $(\mathfrak{g}_{\mathrm{aff}})^\vee$: unique up to $\overset{\mathbb{Z}}{\cancel{\mathbb{C}}}\delta$

$\hat\lambda, \hat\mu$: lifts s.t. $\ell\langle \hat\lambda - \hat\mu, d\rangle = k\ \cancel{\ell}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \; c_2$

Main Conjecture 1

$$H^*(IC(\mathcal{U}_{G,\bar\mu}^{\bar\lambda,\cancel{d}\,k})) = V(\hat\lambda)_{\hat\mu}$$

Rem.

① $\mathbb{V}(\lambda + c\delta)_{\mu + c\delta} \cong \mathbb{V}(\lambda)_{\mu}$

② $\exists$ graded version

LHS : cohomological grading

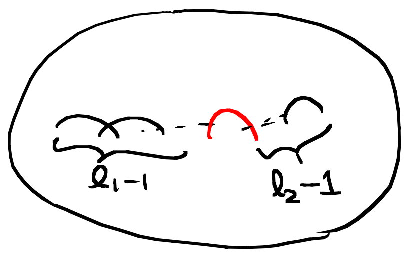
RHS : principal nilpotent

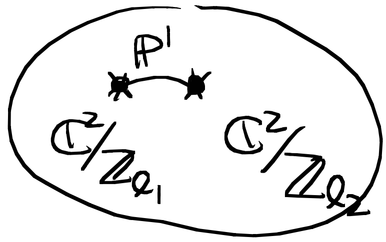
Tensor product

$\ell = \ell_1 + \ell_2$

partial resolution

$$\mathbb{C}^2/\mathbb{Z}_\ell \longleftarrow X_{\ell_1,\ell_2} \longleftarrow$$





$(\ell - 1)$ $\mathbb{P}^1$'s
$\quad\|$
$(\ell_1 - 1) + (\ell_2 - 1) + 1$

Consider Uhlenbeck space on X_{ℓ_1,ℓ_2}

$\mathcal{U}_{G,\mu}^{\lambda_1,\lambda_2,d}$ $\qquad$ $\lambda_1, \lambda_2 : \mathbb{Z}/_{\ell_1}, \mathbb{Z}/_{\ell_2} \longrightarrow G$

level ℓ_1, ℓ_2 weights

Technical conjecture

$\exists$ semismall morphism $\pi : \mathcal{U}^{\lambda_1,\lambda_2,d}_{G,\mu}(X_{\ell_1,\ell_2}) \to \mathcal{U}^{\lambda,d}_{G,\mu}(\mathbb{C}^2)$, $\lambda = \lambda_1+\lambda_2$

Main Conjecture 2

$$\pi_* IC(\mathcal{U}^{\lambda_1,\lambda_2,d}_{G,\mu}(X_{\ell_1,\ell_2})) = \bigoplus IC(\mathcal{U}^{\lambda',d}_{G,\mu})^{\oplus m^{\lambda'}_{\lambda_1,\lambda_2}}$$

$\oplus$ other

with $$\left(V(\lambda_1)\otimes V(\lambda_2)\right)_\mu = \bigoplus_{\lambda'} V(\lambda')_\mu^{\oplus m^{\lambda'}_{\lambda_1,\lambda_2}}$$

Th. conjectures (except graded version of MC1) are true for $G = SL(r)$

$G = SL(r)$ ---- $\mathcal{U}^{\lambda,d}_{G,\mu}$ is an (affine) quiver variety

its IC sheaf was computed

---- related to rep. theory of $\cancel{\widehat{sl}_\ell}$ at level $= r$

$(sl_\ell)_{aff}$

I. Frenkel level-rk duality

$$\widehat{sl}(r)_\ell \longleftrightarrow \widehat{sl}(\ell)_r$$

$\otimes \longleftrightarrow$ branching to $\widehat{sl}(\ell_1)_r \oplus \widehat{sl}(\ell_2)_r$

I develop the theory for the branching in the quiver variety

<u>Rem</u> ① technical advantage for $G = SL(r)$
$\exists$ nice resolution of $\mathcal{U}_{G,\mu}^{\wedge}$

(Gieseker cptification)

② quiver variety generalization to other $\Gamma \subset SL(2)$
$\leftrightarrow$ affine ADE

But the gauge group is $SL(r)$ or $GL(r)$

~~<u>Question</u>. What kind of algebraic structure controls~~
~~e.g. G_{E_8} - instantons on $\mathbb{R}^4/\Gamma_{E_8}$?~~

~~I. Frenkel's joke: Monster ?~~